高瞻計畫_振動學課程
Lecture 3: Single Degree of Freedom Systems (II)

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Outline

- Forced harmonic vibrations
- 2nd order ODEs with forcing term
- Engineering examples
- Vibration isolation: introduction
- Damped response
- Frequency response and resonance behavior
- Simple problems
- Demonstrations

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Part I. Forced Harmonic Vibrations

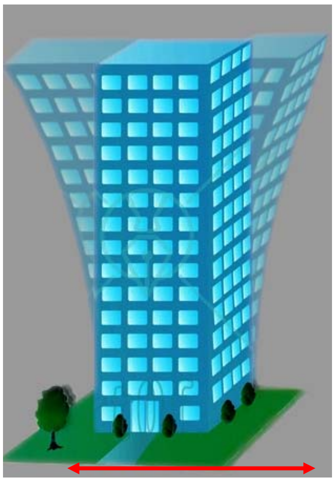
Introduction

- In the previous lecture, we discuss the free response of SDOF systems
- In this lecture, the main thing to be addressed is the behavior of SDOF systems subjected to a harmonic excitation
  - Response
  - Vibration isolation
  - Responance

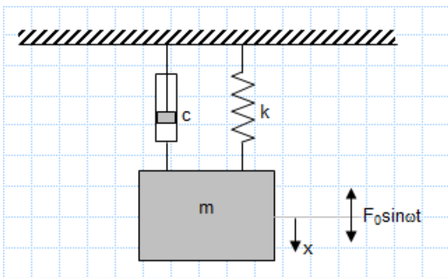
Important Concept of the Previous Lecture

- Mass-spring-damper system
- Newton's method
  - Free body diagrams
- Energy method
  - Rayleigh's method
- Natural frequency
- Damping ratio
- Critical damping

Engineering Problem Statement



Engineering Problem (E.g., Earthquake)



Engineering/Vibration Model

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

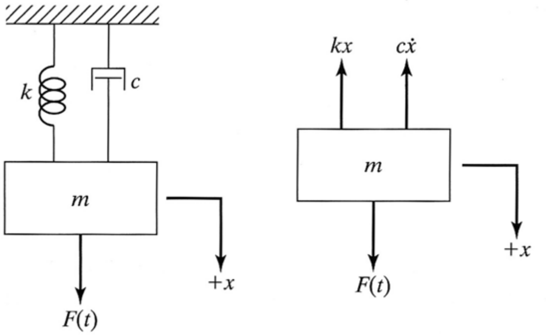
Vibration/Mathematical Model

Introduction

- Forced vibration is the one in which external energy is added to the vibrating system.
- The amplitude of a forced-undamped vibration would increase over time until the mechanism was destroyed.
- The amplitude of a forced-damped vibration will settle to some value where the energy loss per cycle is exactly balanced by the energy gained.

Model Development

$$m\ddot{x} + c\dot{x} + kx = F(t)$$



(a) (b) Free-body diagram

Typical Responses

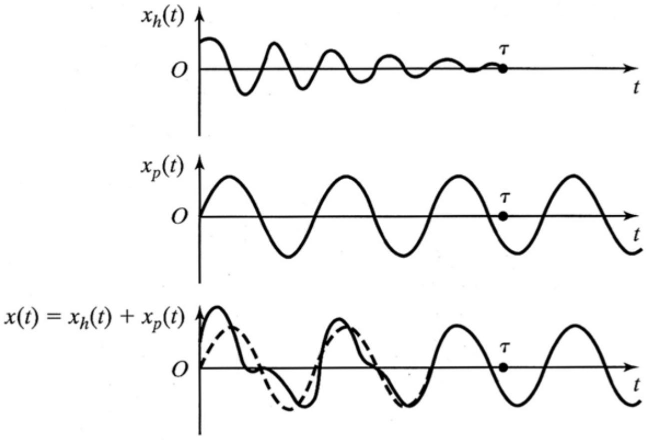


FIGURE 3.2 Homogenous, particular, and general solutions of Eq. (3.1) for an underdamped case.

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Important Concepts

- Distinguish between ω_n and ω
  - ω_n Natural frequency, characteristics of system itself
  - ω Driving frequency, determined by input force
- Two types of responses
  - The complete responses come from two contributions
  - Transient response: will decade to zero as time becomes large
    - Also known as homogeneous or complementary solutions (in ODE)
  - Steady state response: will dominate the system behavior as time becomes large
    - Also known as particular solutions (in ODE)

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Part II. 2nd order ODEs w/ Forcing Term

General Mathematics Development (I)

$$m\ddot{x} + kx = F_0 \cos \omega t$$

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t$$

$$x_p(t) = X \cos \omega t$$

$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

General Mathematics Development (II)

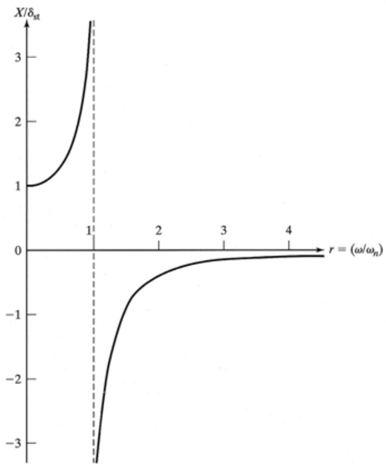
$$C_1 = x_0 - \frac{F_0}{k - m\omega^2}, \qquad C_2 = \frac{\dot{x}_0}{\omega_n}$$

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t$$

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$x_p(t) = -X\cos\omega t$$

$$X = \frac{\delta_{st}}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}$$



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Solution for Under damped Systems

- Frequency response: in and out phase

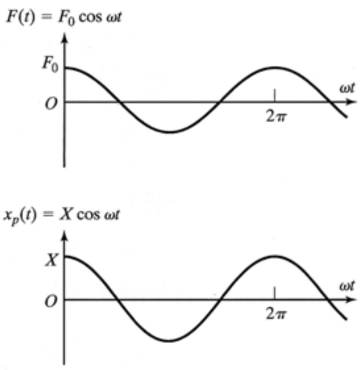

FIGURE 3.4 Harmonic response when $0 < \omega/\omega_n < 1$.

In phase

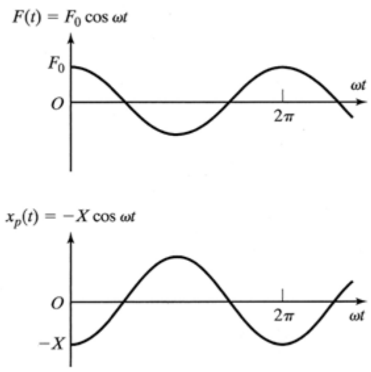

FIGURE 3.5 Harmonic response when $\omega/\omega_n > 1$.

Out phase

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Comparison of free and forced response

- **Sum of two harmonic terms of different frequency**
- **Free response has amplitude and phase effected by forcing function**
- **Our solution is <u>not</u> defined for $\omega_n = \omega$ because it produces division by 0.**
- **If forcing frequency is close to natural frequency the amplitude of *particular* solution is very large**

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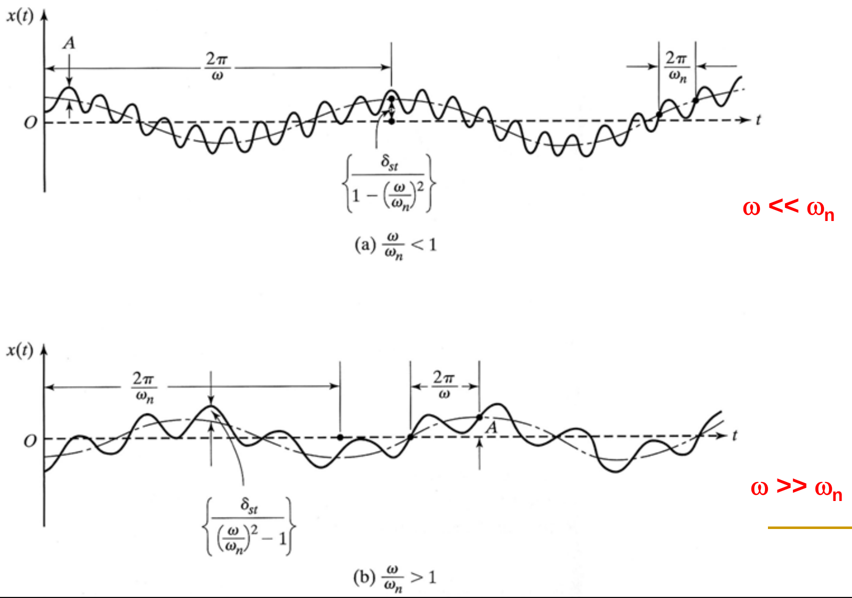
Solution for undamped systems



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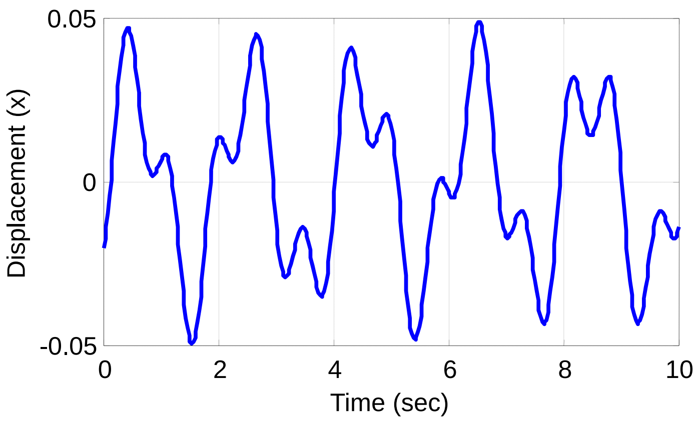
Response for m=100 kg, k=1000 N/m, F=100 N, $\omega = \omega_n$ +5 v_0=0.1m/s and x_0= -0.02 m.

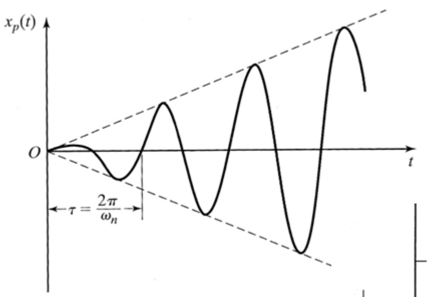


Note the obvious presence of two harmonic signals

Go to code demo

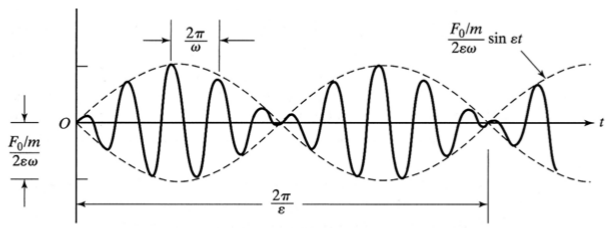
Solution for undamped system

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st}\left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$



$\omega = \omega_n$

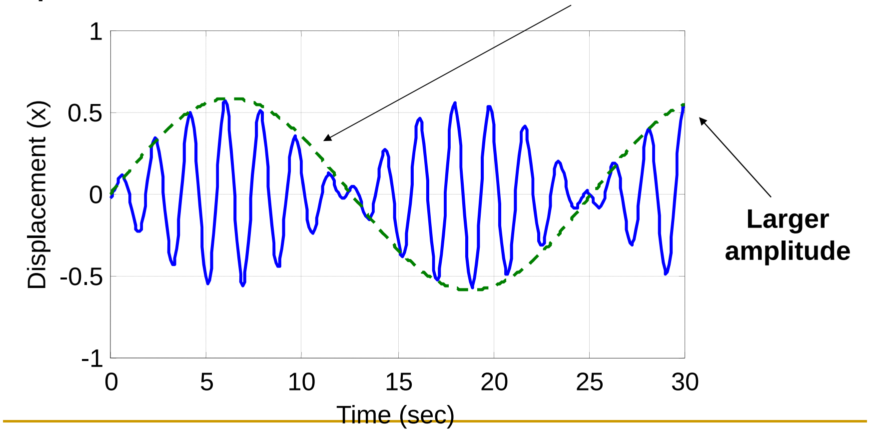
Beating phenomenon
$\omega \sim \omega_n$



What happens when ω is near ω_n?

When the drive frequency and natural frequency are close a beating phenomena occurs

$$\frac{2 f_0}{\omega_n^2 - \omega^2} \sin\left(\frac{\omega_n - \omega}{2} t\right)$$



What happens when ω is ω_n?

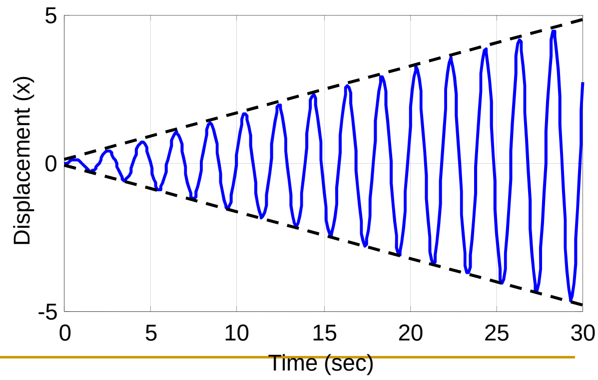
$x_p(t) = tX \sin(\omega t)$

substitute into eq. and solve for X

$$X = \frac{f_0}{2\omega}$$

$$x(t) = A_1 \sin \omega t + A_2 \cos \omega t + \underbrace{\frac{f_0}{2\omega} t \sin(\omega t)}_{\text{grows with out bound}}$$

When the drive frequency and natural frequency are the same the amplitude of the vibration grows without bounds. This is known as a *resonance* condition



Resonance (共振)

The external driving frequency matches one of the system natural frequencies

Part III. Damped Responses

Mathematics (I)

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$x_p(t) = X \cos(\omega t - \phi)$$

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t$$

$$X[(k - m\omega^2)\cos\phi + c\omega \sin\phi] = F_0$$
$$X[(k - m\omega^2)\sin\phi - c\omega \cos\phi] = 0$$

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

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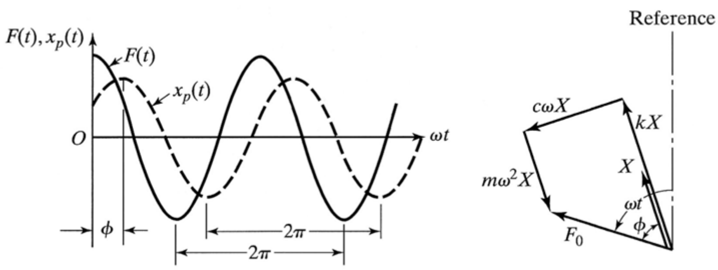
Mathematics (II)



$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$r \equiv \frac{\omega}{\omega_n}$$

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

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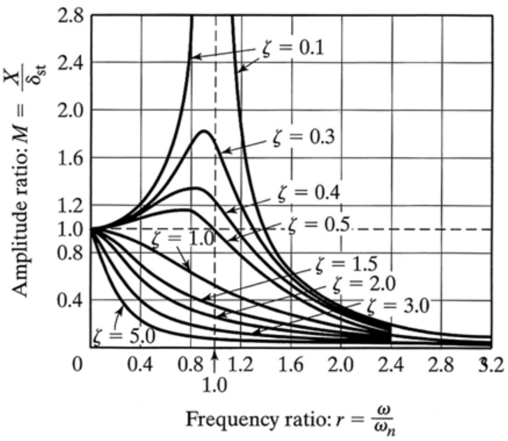
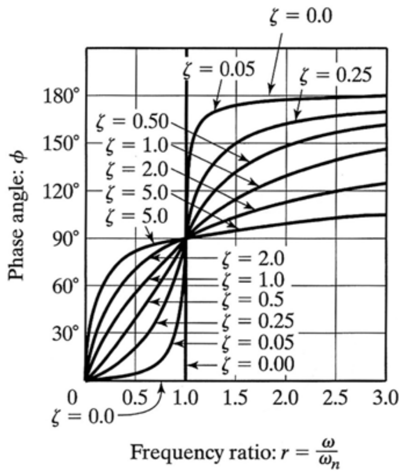
Frequency Responses

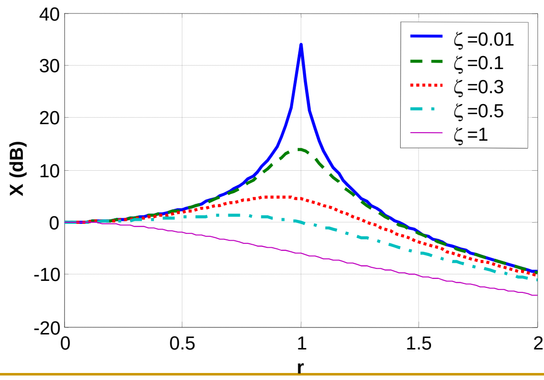




Magnitude plot

- **Resonance is close to *r* = 1**
- **For ζ = 0, *r* =1 defines resonance**
- **As ζ grows resonance moves *r* <1**
- **The exact value of *r*, can be found from differentiating the magnitude**

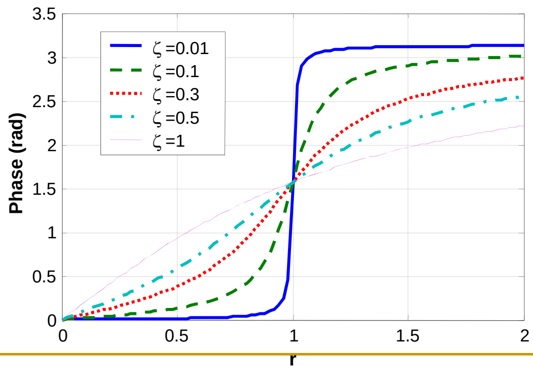
$$X = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



Phase plot

$$\theta = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2} \right)$$

- **Resonance occurs at $\phi = \pi/2$**
- **The phase changes more rapidly when the damping is small**
- **From low to high values of *r* the phase always changes by 180^0 or π radians**



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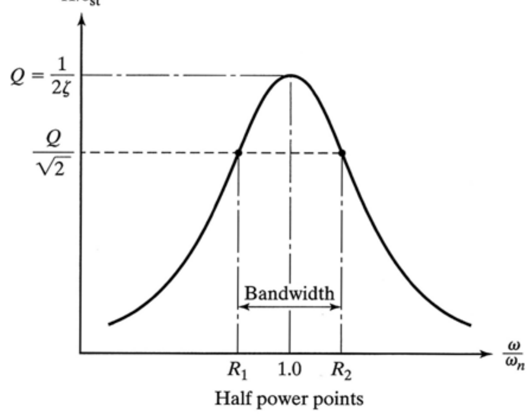
Quality Factor and Bandwidth

$$\left(\frac{X}{\delta_{st}}\right)_{max} \simeq \left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} = Q$$

$$\Delta W = \pi c \omega X^2$$



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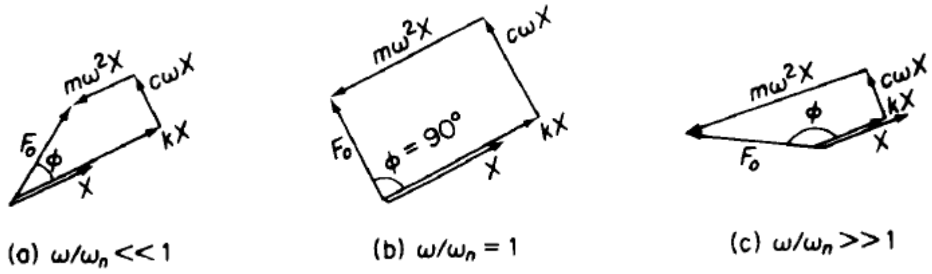
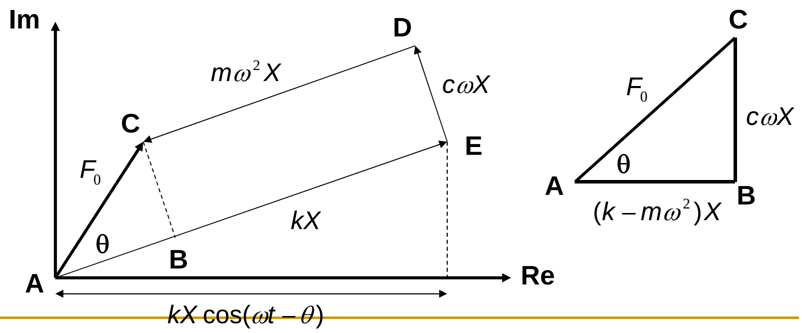
Phasor Representation of Vibration Equations

- Spring force, damping force, and inertia force have 90◦phase differences.
- At resonance, if damping is small, X/F_o can be huge



Geometric Approach

- **Position, velocity and acceleration phase shifted each by $\pi/2$**
- **Therefore write each as a vector**
- **Compute X in terms of F_0 via vector addition**



Using vector addition on the diagram:

$$F_0^2 = (k - m\omega^2)^2 X^2 + (c\omega)^2 X^2$$

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

At resonance:

$$\theta = \frac{\pi}{2}, \qquad X = \frac{F_0}{c\omega}$$

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Rotating Unbalance

The effects of unbalance is a common problem in vibrating systems.

Consider a one dimensional system with an unbalance represented by an eccentric mass, m, with offset, e, rotating at some speed, ω, as shown

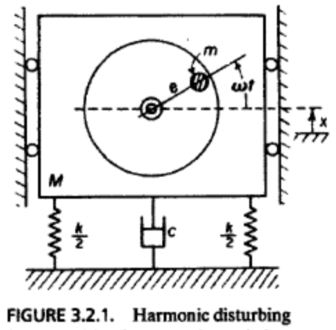

FIGURE 3.2.1. Harmonic disturbing force resulting from rotating unbalance.

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Rotating Unbalance

- Equation of Motion

$$(M-m)\ddot{x}+m\frac{d^2}{dt^2}(x+e\sin\omega t)=-kx-c\dot{x}$$

$$M\ddot{x}+c\dot{x}+kx=\left(me\omega^2\right)\sin\omega t$$

- Solution

$$\frac{M}{m}\frac{X}{e}=\frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}}$$

$$\phi=\tan^{-1}\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

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Rotating Unbalance

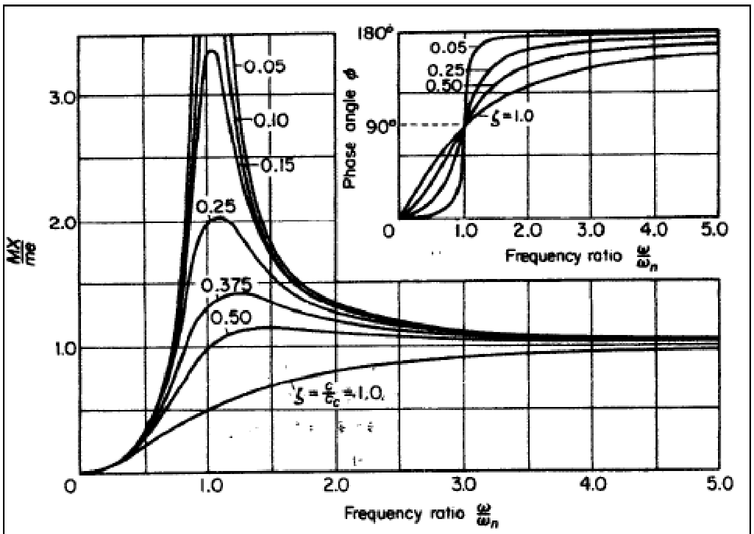

FIGURE 3.2.2. Plot of Eqs. (3.2.4) and (3.2.5) for forced vibration with rotating unbalance.

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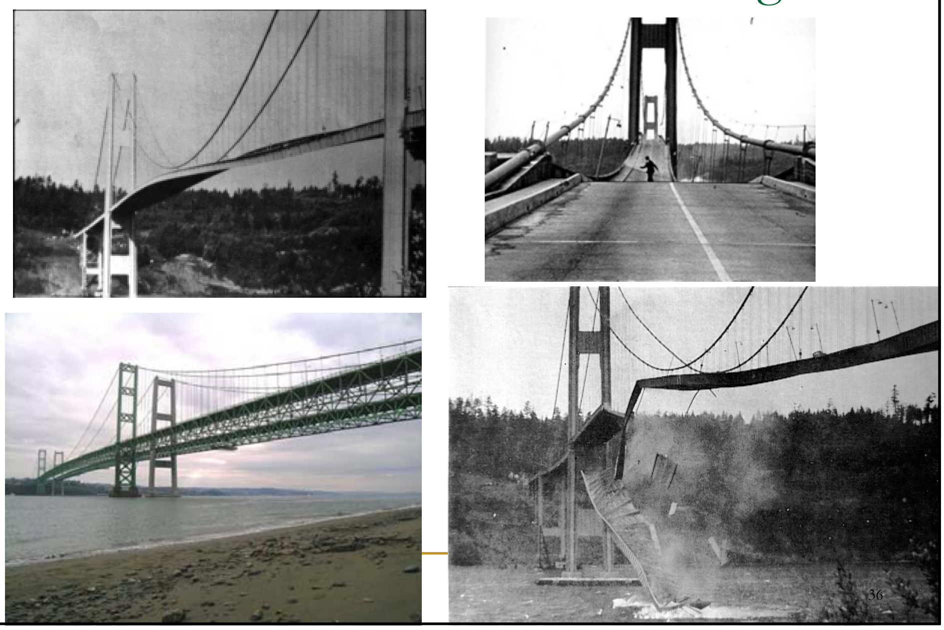
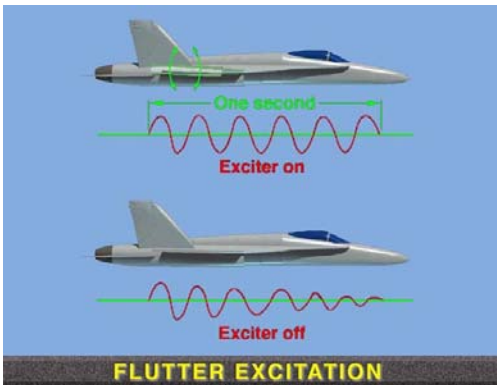
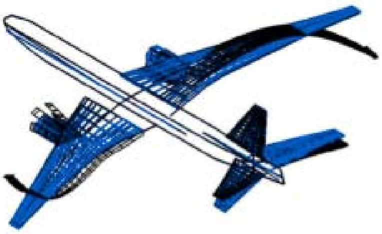
Part IV. Engineering Examples

Failure of Tachoma Narrow Bridge



Flutter of Airplanes





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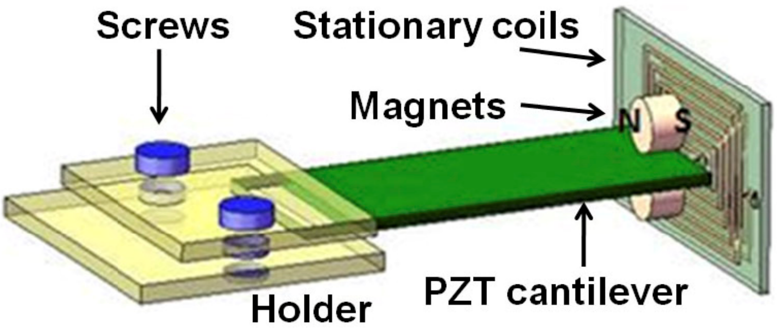
Vibration Energy Harvester



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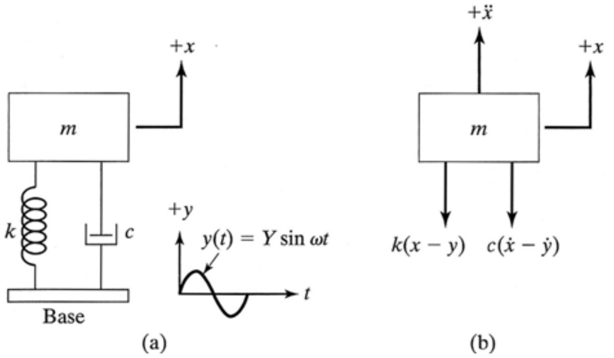
Part V. Vibration Isolation: Introduction

Introduction

- Vibration isolation is (perhaps) the most important application of vibration course in industry
- Goals:
  - Attenuate or isolate base vibration transmitted to the work piece
  - Attenuate the seismic damage on buildings
  - Noise reduction
  - Reducing motion –induced vibration of flexible structure during operation

Base Excitation

- Study the influence of base motion Y (force or displacement) on the top mass X
  - E.g., Y=floor motion, X=optical table



Base Excitation

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY \sin \omega t + c\omega Y \cos \omega t$$
$$= A \sin (\omega t - \alpha)$$

$$x_p(t) = \frac{Y\sqrt{k^2 + (c\omega)^2}}{\left[(k - m\omega^2)^2 + (c\omega)^2\right]^{1/2}} \sin (\omega t - \phi_1 - \alpha)$$

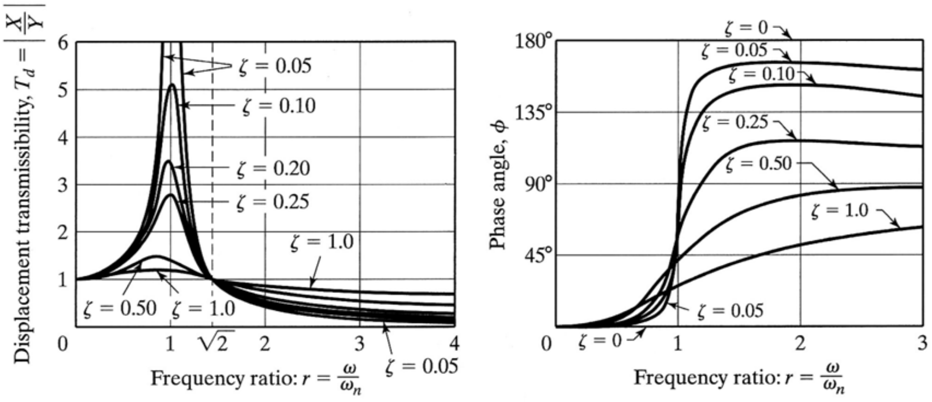
$$x_p(t) = X \sin (\omega t - \phi)$$

$$\frac{X}{Y} = \left[\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}\right]^{1/2} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$\phi = \tan^{-1}\left[\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2}\right] = \tan^{-1}\left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right]$$

Base Excitation

- Transmissibility = X/Y
  - Attenuation ratio of the base motion through the system



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Force Transmitted

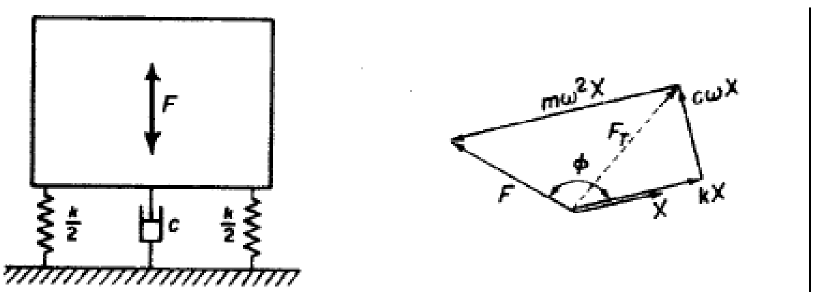

FIGURE 3.6.1. Disturbing force transmitted through springs and damper.

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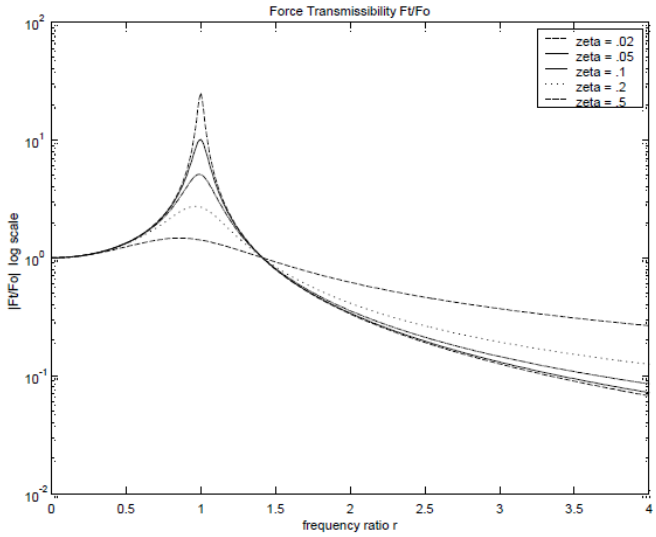
Force Transmissibility



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Important Insights

- Base vibration can be attenuated when r>√2
- Since r=ω/ωn, this implies that a smaller ωn can achieve a better isolation range
- However, since ωn = √(k/m), a smaller ωn → a smaller k
  - Problem of DC disturbance rejection
  - This is also the reason why your optical table is quite compliant if you try to apply a low frequency load
- May need to use active vibration for low frequency disturbance

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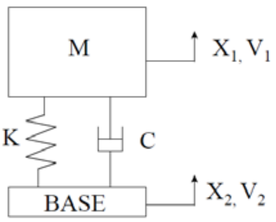
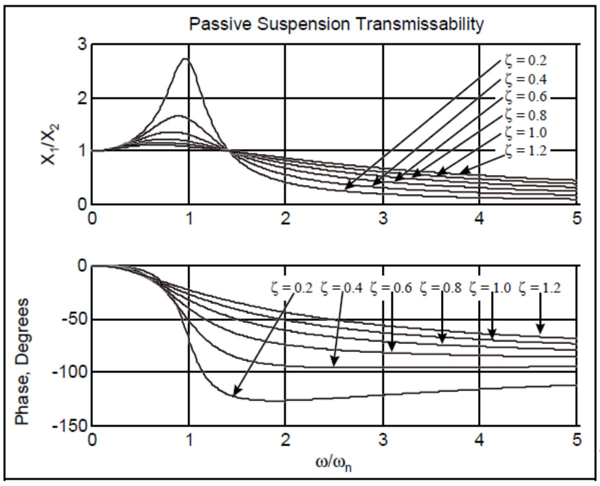
Passive Damper

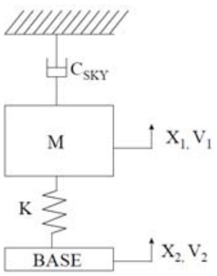
$$\frac{X_1}{X_2} = \frac{1 + j2\zeta_P(\frac{\omega}{\omega_n})}{1 - (\frac{\omega}{\omega_n})^2 + j2\zeta_P(\frac{\omega}{\omega_n})}$$

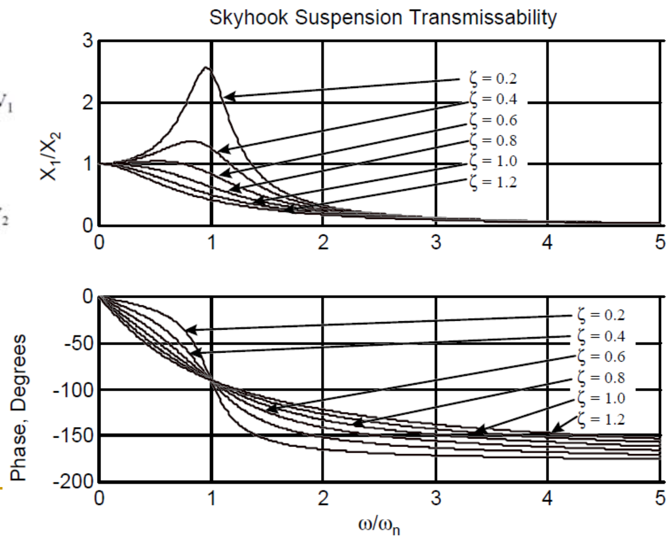




Skyhook Damper

$$\frac{X_1}{X_2} = \frac{1}{1 - (\frac{\omega}{\omega_n})^2 + j2\zeta_{SKY}(\frac{\omega}{\omega_n})}$$





Vibration Measurement Instruments

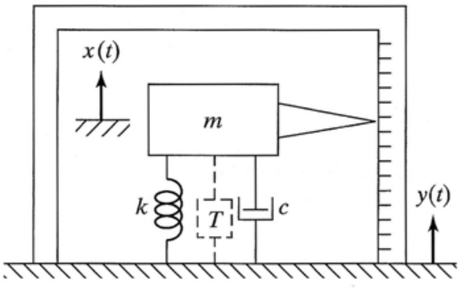
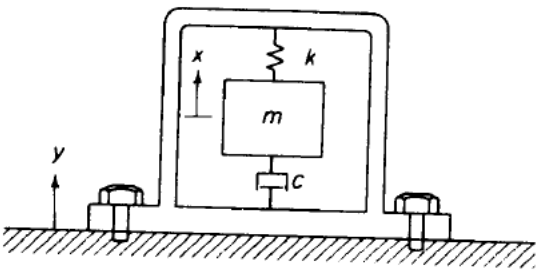


FIGURE 10.9 Seismic instrument.

Vibration Measurement Instruments

- Goal: measure the status of y by means of x
- Vibrometers:
  - Measure y by x
- Accelerometers:
  - Measure $\ddot{y}$ by x



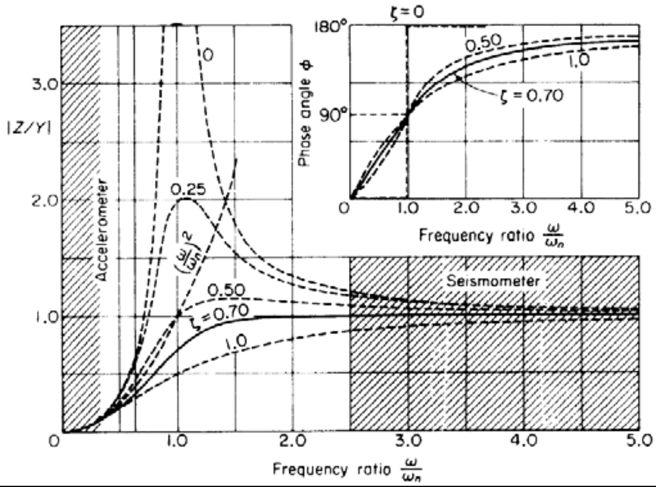
$$m\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y)$$

$$z = (x - y)$$

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin \omega t$$

Vibrometers: High Frequency Devices

$$Z = \frac{m\omega^2 Y}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{Y\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$



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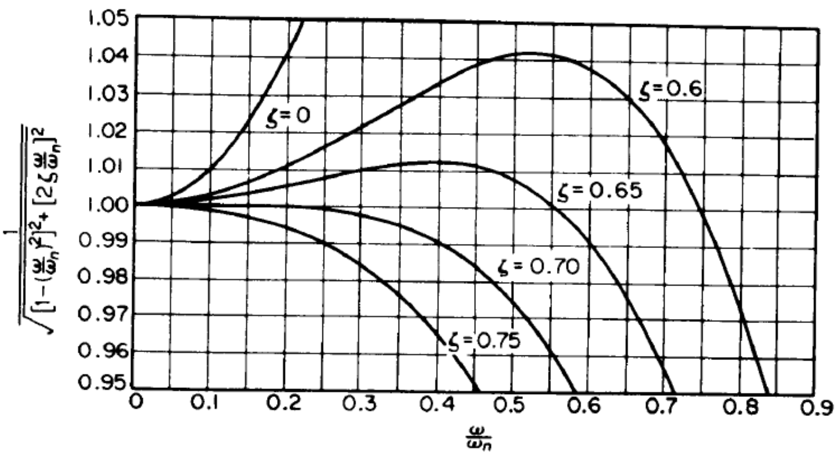
Accelerometers: Low Frequency Devices

- Using $Z/\omega^2 Y$ as the measurement index



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Working Range of Vibration Measurement Devices

- The flat regime of their FRF
- For vibrometer in previous case, $\omega/\omega_n > 4$
  - This implies that a smaller ω_n is desired
  - Soft structure
- For accelerometer in previous case, $\omega/\omega_n < 0.1$
  - This implies that a larger ω_n is desired
  - Stiff structure

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Part VI. Frequency Response and Resonance Behavior

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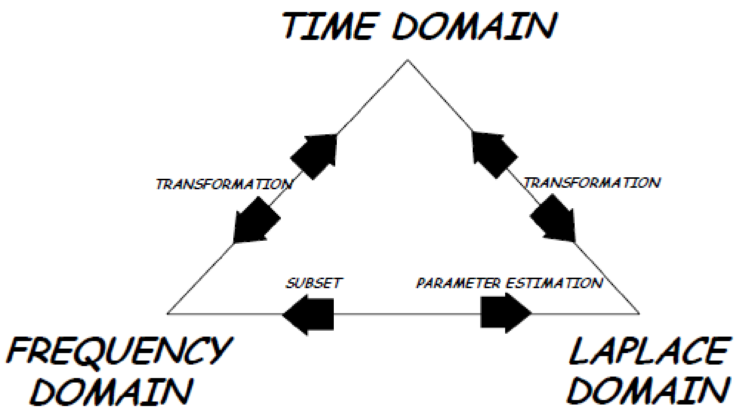
Solution Domains



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Introduction

- Frequency responses functions (FRF) correlate the behavior of vibration systems w.r.t. different input signals
  - Response magnitude and phase delay
  - Equivalent to the Bode diagrams in control engineering
  - Transfer functions
- Major information obtained from FRF
  - Sensitivity, bandwidth, decay slope, …

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Methods to Obtain FRF

- Swept sine test
  - Apply a single sine input $F= F_0\sin\omega t$ and obtain the response $X=X\sin(\omega t+\phi)$
  - Then plot the magnitude plot (i.e., X/F_0 vs.ω) and phase plot (i..e, ϕ vs. ω)
  - Change ω and repeat the procedure to obtain sufficient data
- Impact test
  - Apply an “impact” force
  - Perform FFT to convert the obtained data to its frequency contents
  - Repeat and average

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Resonance Behavior

- The input and output has a 90^o phase difference
  - Energy continuously feeds into the system after each cycle
  - Results larger vibration amplitude
  - Bad for structural engineering
  - Might be good for actuating system design
  - Amplification ratio depends on the system damping ratio

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Instability Issues

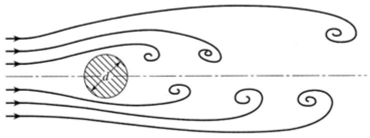
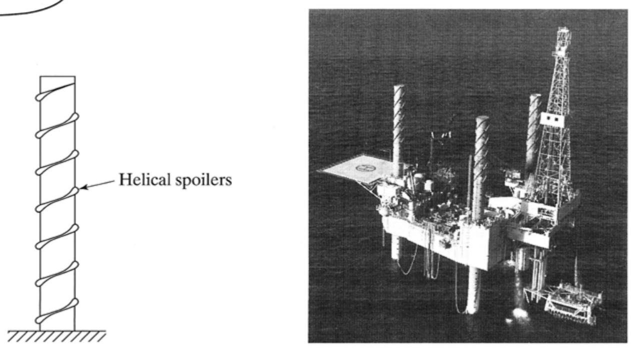


FIGURE 3.28 Fluid flow past a cylinder.



(a) (b)

FIGURE 3.30 Helical spoilers. (Photo courtesy of Bethlehem Steel Corporation).

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Part VII. Simple Problems

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Problem 1. Plate support a pump (Rao 3.1)

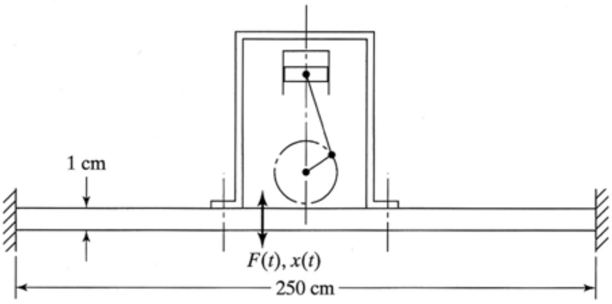

FIGURE 3.9 Plate supporting an unbalanced pump.

A reciprocating pump, having a mass of 68 kg, is mounted at the middle of a steel plate of thickness 1 cm, width 50 cm, and length 250 cm, clamped along two edges as shown in Fig. 3.9. During operation of the pump, the plate is subjected to a harmonic force, $F(t) = 220 \cos 62.832t$ N. Find the amplitude of vibration of the plate.

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Problem 2. Vehicle move on a rough road (Rao 3.3)

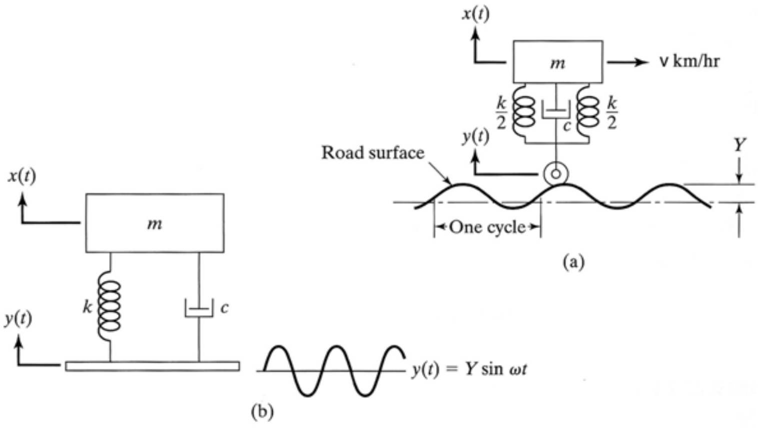

FIGURE 3.18 Vehicle moving over a rough road.

Figure 3.18 shows a simple model of a motor vehicle that can vibrate in the vertical direction while traveling over a rough road. The vehicle has a mass of 1200 kg. The suspension system has a spring constant of 400 kN/m and a damping ratio of $\zeta = 0.5$. If the vehicle speed is 20 km/hr, determine the displacement amplitude of the vehicle. The road surface varies sinusoidally with an amplitude of $Y = 0.05$ m and a wavelength of 6 m.

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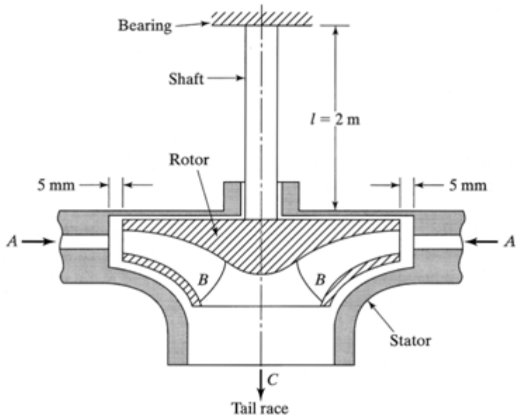
Problem 3 Francis Water Turbine (Rao 3.5)



The schematic diagram of a Francis water turbine is shown in Fig. 3.20 in which water flows from *A* into the blades *B* and down into the tail race *C*. The rotor has a mass of 250 kg and an unbalance (*me*) of 5 kg-mm. The radial clearance between the rotor and the stator is 5 mm. The turbine operates in the speed range 600 to 6000 rpm. The steel shaft carrying the rotor can be assumed to be clamped at the bearings. Determine the diameter of the shaft so that the rotor is always clear of the stator at all the operating speeds of the turbine. Assume damping to be negligible.

Part VIII. Youtube Demonstrations